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Multiresonant control of two-dimensional dynamical systems

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It is shown that many two degree of freedom (2D) nonlinear dynamical systems can be controlled by continuous phase-locking (double autoresonance) between the two canonical angle variables of the system and two independent external oscillating perturbations having slowly varying frequencies. Conditions for stability of the 2D autoresonance and classification of systems with doubly autoresonant solutions in the vicinity of a stable equilibrium are outlined in terms of the Hessian matrix elements of the unperturbed system. The doubly autoresonant states in a generic, driven 2D system can be accessed by starting in equilibrium and simultaneous passage through two linear resonances in the system, provided that the driving amplitudes exceed a threshold scaling as $\alpha^{3/4}$, α being the characteristic chirp rate of the driving frequencies. The formation of nearly periodic trajectories in linearly nondegenerate, 2D driven systems with a single stable equilibrium is suggested as an application. Examples of autoresonant excitation and formation of nearly periodic states in other types of driven systems are presented, including a three-particle Toda chain, a particle in a 2D double-well potential, and a 3D oscillator.

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I. INTRODUCTION

Many physical applications require formation of a state of a n-dimensional dynamical system having well defined properties. For example, the Einstein-Brillouin-Keller (EBK) semiclassical quantization requires a state with prescribed (quantized) values of the canonical action variables of the system (see, for example, Ref. [1], and references therein). This goal can be successfully achieved by using the adiabatic switching approach (for an excellent review of the method see Ref. [2]). This method uses the preservation of the actions under adiabatic variation of the system's parameters. Thus, by starting from a simple system with known values of the action variables one can adiabatically transform it into a system of interest having the same values of the actions by varying a single parameter. In other applications, one is interested in finding orbits in integrable multidimensional systems with a given ratio of the characteristic frequencies. In particular, periodic orbits with a rational ratio of the frequencies are important in both semiclassical description of dynamical systems [3] and for understanding nontrivial atomic spectra [4,5]. Unfortunately, the adiabatic switching approach is generally inapplicable in finding periodic orbits because the adiabatic preservation of the action variables does not imply (except in particular applications [6]) preservation of the periodicity of the trajectory in the system.

In this work, we study the multifrequency autoresonance in multidimensional systems and propose to use this phenomenon in controlling the characteristic frequencies (as opposed to actions in the adiabatic switching method) of dynamical systems. The single frequency autoresonance is currently well understood and serves as a basis in many applications, such as particle accelerators [7,8], excitation of atoms [9,10] and molecules [11,12], and formation of nonlinear waves [13]. The idea behind all these applications is the salient feature of many nonlinear systems, at certain conditions, to adiabatically preserve the phase locking with driving perturbations, so that the driven system remains in an approximate resonance with the drive, even if the driving frequency and/or other parameters vary slowly in time. This preservation of resonance (autoresonance) is due to the automatic and continuous self-adjustment of the driven system to changing conditions. The autoresonance is an intrinsically nonlinear phenomenon and, in addition to the adiabaticity, one requires a sufficient nonlinearity in the system for continuous phase locking. Recently, the idea of single-frequency autoresonance was extended to multifrequency autoresonance in several integrable dynamical [14] and nonlinear wave [15] systems. In these cases one applies a superposition of driving waves and oscillations, each coupled to a particular degree of freedom. Then, passage through linear resonances yields initial multiphase locking, which is preserved due to autoresonance at a later stage, yielding efficient excitation and control of the desired waves or oscillations. We propose to apply similar ideas to a more general class of multidimensional dynamical systems. For simplicity, we will present the details of the theory for the two degree of freedom (2D) case, higher dimensionality (see a numerical 3D example at the end of this work) can be dealt with similarly. We will study a driven two-dimensional oscillator governed by the Hamiltonian

$$H=\frac{1}{2}(p_1^2+p_2^2)+V(q_1,q_2,\lambda)+f(q_1,q_2,t), \tag{1}$$

where $q_{1,2}$ and $p_{1,2}$ are the canonical coordinates and momenta, $V(q_1,q_2,\lambda)$ describes the potential energy, λ is a parameter (it may slowly vary in time), while a small driving term $f=\epsilon_1 q_1\cos[\varphi_1(t)]+\epsilon_2 q_2\cos[\varphi_2(t)]$ represents the effect of a superposition of two external, spatially uniform, but oscillating forces $\epsilon_{1,2}\cos(\varphi_{1,2})$ in $q_{1,2}$ directions, respectively. We will assume that all variables and parameters in Eq. (1) are dimensionless and that the driving frequencies

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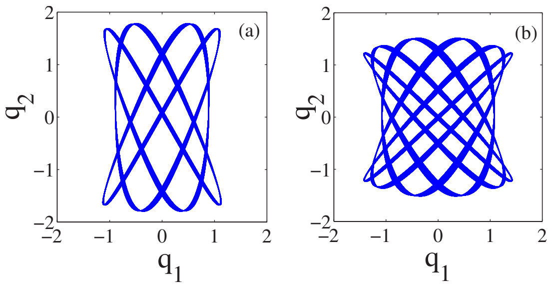

FIG. 1. (Color online) Nearly periodic orbits in a driven 2D quadric potential system. The driving frequencies $\omega_{1,2}$ were swept adiabatically through the linear frequencies, reaching final values having a desired rational ratio (a) $\omega_1:\omega_2=3:4$ and (b) $\omega_1:\omega_2=5:6$. In both examples, the linear and nonlinear terms in the Hamiltonian are of the same order. 200 cycles of driven oscillations are shown.

$\omega_{1,2}(t)=\varphi_{1,2}$ are slow functions of time. As an application of the autoresonant ideas, we will focus on the possibility of autoresonant formation of periodic trajectories in the process of evolution, by choosing a desired final rational ratio $\omega_1:\omega_2=m:n$ of the adiabatically varying driving frequencies. Numerical examples in Fig. 1 illustrate our autoresonance approach, showing nearly periodic orbits with different frequency ratios in the case of adiabatically driven quadric oscillator $V=\frac{1}{2}(\nu_1^2q_1^2+\nu_2^2q_2^2)+\gamma_1q_1^4+\gamma_2q_2^4+\delta q_1^2q_2^2$, with $\{\nu_1,\nu_2,\gamma_1,\gamma_2,\delta\}=\{0.8,1,0.21,0.3,0.32\}$. Studying excitation dynamics and stability of the autoresonant processes involved in this and other two-dimensional systems comprises the main goal of the present work.

The scope of the presentation will be as follows. The existence and stability of two-frequency autoresonance in our driven system will be discussed in Sec. II. We will assume existence of a *local* approximate action-angle (AA) representation in studying this problem. Section III will deal with the general problem of two-dimensional autoresonance in the vicinity of equilibrium. Passage through the linear resonances in this weakly nonlinear problem will yield a convenient approach to the initial phase locking and autoresonance in the system, a necessary step for entering a stable, strongly nonlinear autoresonance regime. In Sec. IV, we will present a number of numerical examples of autoresonant formation and control of nearly closed orbits in more complex systems. This will include the driven, three-particle, integrable and nonintegrable Toda systems, the formation of periodic orbits in a double-well potential by using a combination of autoresonance and the adiabatic switching approach, and the excitation of a 3D periodic state of three coupled nonlinear oscillators.

II. AUTORESONANCE OF TWO DEGREES OF FREEDOM

Consider a driven two-dimensional dynamical system governed by Hamiltonian (1). Suppose also that there exist a nearly regular region in phase space [19], such that one can rewrite the Hamiltonian in the form

$$H=H_0(\mathbf{I},\lambda)+H_1(\mathbf{I},\theta,\lambda)+f(\mathbf{I},\theta,t), \quad (2)$$

where $I_{1,2}$ and $\theta_{1,2}$ are the AA variables of the unperturbed Hamiltonian (H_0), and the perturbations $|H_1|,|f|$ are much smaller than $|H_0|$. In this region, H_1 is a periodic function of the θ's. By expanding in Fourier series, $q_i=\Sigma_{k,l}A_{k,l}^{(i)}\times(I_1,I_2,\lambda)e^{i(k\theta_1+l\theta_2)}$, $i=1,2$ and assuming the continuing doubly fundamental resonance in the system $\Omega_{1,2}\equiv\partial H_0/\partial I_{1,2}\approx\omega_{1,2}(t)$ we will replace the driving term f in Eq. (2) by its double resonance approximation [17]

$$f\approx\epsilon_1a_1\cos\psi_1+\epsilon_2a_2\cos\psi_2, \quad (3)$$

where $a_1=\frac{1}{2}|A_{1,0}^{(1)}(\mathbf{I})|$, $\phi_1=\mathrm{Arg}(A_{1,0}^{(1)})$, $a_2=\frac{1}{2}|A_{0,1}^{(2)}(\mathbf{I})|$, $\phi_1=\mathrm{Arg}(A_{0,1}^{(2)})$, and $\psi_i=\theta_i-\varphi_i+\phi_i$ are the phase mismatches between the canonical angles and the phases of the corresponding drives. Similarly, we expand $H_1=\Sigma_{k,l}h_{k,l}(I_1,I_2,\lambda)e^{i(k\theta_1+l\theta_2)}$. The resonant contribution in H_1 comes from the terms $k=n,l=m$ in this series such that $n\Omega_1\approx m\Omega_2$. We will assume that the coefficients $h_{m,n}$ of these terms are sufficiently small compared to those in Eq. (3) for completely neglecting the effect of H_1 in the driven dynamics (see Sec. IV C). Then, the evolution of the actions and phase mismatches is described by

$$\dot I_i=\epsilon_ia_i\sin\psi_i, \quad (4)$$

$$\dot\psi_i=\Omega_i(\mathbf{I},\lambda)-\omega_i(t)+\dot\phi_i+\sum_{m=1,2}\epsilon_m\frac{\partial a_m}{\partial I_i}\cos\psi_m. \quad (5)$$

Our goal is to show that at certain conditions, this system yields autoresonant evolution, i.e., slowly evolving solution (governed approximately by H_0), which stays in a continuing double resonance with the driving oscillations, despite variation of system's parameters $\omega_{1,2}$ and λ.

The theory of the double autoresonance in the system proceeds similarly to the conventional theory of a single nonlinear resonance [16]. We assume that $\dot\phi_i$ and the terms with ϵ_m in Eqs. (5) are small compared to $\Omega_i(\mathbf{I},\lambda)-\omega_i(t)$, which are of $O(\epsilon^{1/2})$ (see below) and approximate our system by

$$\dot I_i=\epsilon_ia_i\sin\psi_i,\dot\psi_i=\Omega_i(\mathbf{I},\lambda)-\omega_i(t). \quad (6)$$

We seek solutions of these equations in the form $I_i=\bar I_i+\delta I_i$, $\psi_i=\bar\psi_i+\delta\psi_i$, where δI_i and $\delta\psi_i$ are the oscillatory components, while $\bar I_i$ and $\bar\psi_i$ are the slowly evolving averages of the corresponding variables. We also assume that $\bar\psi_i$ is close to either 0 or π, so that $\sin\bar\psi_i\approx s_i\bar\psi_i'$ where $s_i=\pm1$ and $\bar\psi_i'$ is either $\bar\psi_i$ or $\bar\psi_i-\pi$ for $s=1$ or -1, respectively. Furthermore, we view δI_i as small and scaling as $\sqrt{\epsilon}$ with the driving amplitudes, while no special scaling with ϵ is assumed for $\delta\psi_i$, but still $\delta\psi_i$ are viewed as small, allowing linearization $\sin(\delta\psi_i)\approx\delta\psi_i$ in the following. Next, we linearize and separate the oscillatory and average parts in Eqs. (4) and (5), yielding, for the oscillatory components,

$$\delta\dot I_i=\epsilon_i\bar a_is_i\delta\psi_i, \quad (7)$$

$$\delta\dot\psi_i=\sum_{j=1,2}\beta_{i,j}\delta I_j, \quad (8)$$

where

$$\beta_{i,j} = \frac{\partial \Omega_i}{\partial \bar{I}_j} = \frac{\partial H_0}{\partial \bar{I}_i \partial \bar{I}_j} \tag{9}$$

is the Hessian associated with H_0 and $\bar{a}_i$, $\beta_{i,j}$ are evaluated at averaged values of the actions. Similarly, the averaged components evolve via

$$\dot{\bar{I}}_i = \epsilon_i \bar{a}_i s_i \bar{\psi}_i', \tag{10}$$

$$\dot{\bar{\psi}}_i' = \bar{\Omega}_i(\bar{\mathbf{I}},\lambda) - \omega_i(t). \tag{11}$$

At this stage, we seek a quasi-steady state solution for the averaged actions $\bar{I}_{1,2}$ given by the adiabatic double resonance conditions

$$\Omega_i(\bar{I}_1,\bar{I}_2,\lambda) = \omega_i(t). \tag{12}$$

Then, by differentiation and use of Eq. (10)

$$\sum_j \beta_{i,j}\epsilon_j \bar{a}_j s_j \bar{\psi}_j' = \dot{\omega}_i - \frac{\partial \Omega_i}{\partial \lambda}\dot{\lambda} \equiv \alpha_i \tag{13}$$

or, in matrix form,

$$\mathbf{B} \cdot \bar{\psi}' = \alpha, \tag{14}$$

where

$$\mathbf{B} \equiv \beta \begin{pmatrix} \epsilon_1 \bar{a}_1 s_1 & 0 \\ 0 & \epsilon_2 \bar{a}_2 s_2 \end{pmatrix}. \tag{15}$$

Thus,

$$\bar{\psi}' = \begin{pmatrix} \dfrac{1}{\epsilon_1 a_1 s_1} & 0 \\ 0 & \dfrac{1}{\epsilon_2 a_2 s_2} \end{pmatrix} \beta^{-1} \cdot \alpha, \tag{16}$$

assuming, of course,

$$D \equiv \det \beta \neq 0. \tag{17}$$

Furthermore, our assumption of the smallness of the solution $\bar{\psi}' \ll 1$ requires relative smallness of the variation rates α_i as compared to the driving amplitudes ϵ_i. Also, since the Hessian β involves second order derivatives of the Hamiltonian with respect to the actions (β=0 in linear problems), the smallness of $\bar{\psi}'$ requires a sufficient degree of nonlinearity of the problem.

Next, we proceed to the oscillating components described by Eqs. (7) and (8), i.e., consider the problem of stability of the multidimensional autoresonance. We fix the slow time variation in the coefficients in these equations (this is a local WKB-type analysis), differentiate Eqs. (8) in time, and use Eqs. (7) in the result, yielding

$$\delta\ddot{\psi} = \mathbf{B} \cdot \delta\psi. \tag{18}$$

We seek solutions of this system in the form $\delta\psi_i \sim e^{i\kappa_i t}$ in which case $-\kappa_{1,2}^2$ must be eigenvalues of $\mathbf{B}$, i.e.,

$$\kappa_{1,2}^2 = \frac{1}{2}[Q \pm \sqrt{Q^2 - R}],$$

where $Q=s_1b_{11}+s_2b_{22}$, $R=4s_1s_2(b_{11}b_{22}-b_{12}b_{21})$, and $b_{ij}=\epsilon_i\bar{a}_i\beta_{ij}$. All $\kappa_{1,2}^2$ must be real and positive for stability ($\kappa_{1,2}^2 \neq 0$ since $R=4s_1s_2\epsilon_1\epsilon_2\bar{a}_1\bar{a}_2 D \neq 0$). Now, we consider two possibilities (a) $D>0$ and (b) $D<0$, separately. In case (a), $s_1s_2>0$ for stability. In case (b), in contrast, simple analysis shows that stability is guaranteed only if $s_1s_2<0$ and $|b_{11}+b_{22}|>2\sqrt{|b_{12}b_{21}|}=2|\beta_{12}|\sqrt{\epsilon_1\epsilon_2\bar{a}_1\bar{a}_1}$. By defining $x=\frac{\beta_{11}}{\beta_{12}}$, $y=\frac{\beta_{22}}{\beta_{12}}$, and $r=\sqrt{\frac{\epsilon_2\bar{a}_2}{\epsilon_1\bar{a}_1}}$, we can replace $D<0$ condition by $xy<1$, while the second condition for case (b) becomes

$$\left|\frac{x}{r} + ry\right| > 2. \tag{19}$$

Since r can be varied by choosing the ratio between the external force amplitudes ϵ_2/ϵ_1, stability of the autoresonance can be guaranteed for any values of x and y (except for cases with $xy=1$, and $x=y=0$). Also, since b_{ij} are of $O(\epsilon)$, the characteristic frequencies of the autoresonant oscillations scale as $\kappa_{i,j} \sim \sqrt{\epsilon}$, while integrating Eqs. (7) we obtain the scaling $\delta I_{i,j} \sim \sqrt{\epsilon}$, as assumed previously. Finally, for the validity of our local WKB-type analysis we require $\mu=\max(|\dot{P}/P|)$, characterizing the slowness of variation of the set $P=\{\omega_1,\omega_1,\lambda\}$, to satisfy the adiabaticity condition, $\mu \ll \min \kappa_i$. This completes our analysis of existence and stability of the autoresonant solution and we proceed to the problem of capturing our system in autoresonance by starting in equilibrium and passing through the linear frequencies in the system

III. DOUBLE AUTORESONANCE IN THE VICINITY OF EQUILIBRIUM

We describe our system in the vicinity of equilibrium ($I_{1,2}$=0) by an approximate, weakly nonlinear Hamiltonian

$$H = \nu_1 I_1 + \nu_2 I_2 + \frac{1}{2}aI_1^2 + bI_1I_2 + \frac{1}{2}cI_2^2 + f(\mathbf{I},\theta,t), \tag{20}$$

where $a=\beta_{11}$, $b=\beta_{12}$, $c=\beta_{22}$ are evaluated at the equilibrium and ν_i are the linear frequencies in the problem. This lowest nonlinear order Hamiltonian can be conveniently calculated by using the canonical perturbation theory [18,19], provided the linear limit is nondegenerate, i.e., $\nu_1 \neq \nu_2$, which is assumed in the following. As in the fully nonlinear case (see Sec. II), we use double resonance approximation in the driving term in Eq. (20). Consequently, and because of the smallness of the driving amplitude, we substitute the linear approximation $q_i=\sqrt{\frac{2I_i}{\nu_i}}\cos\theta_i$ in the driving term in the Hamiltonian [see Eq. (3)], yielding resonant driving contribution

$$f(\mathbf{I},\theta,t) = \epsilon_1\sqrt{\frac{I_1}{2\nu_1}}\cos\psi_1 + \epsilon_2\sqrt{\frac{I_2}{2\nu_2}}\cos\psi_2. \tag{21}$$

In studying the problem of capture into double autoresonance, we consider simultaneous passage through the linear resonances in the system, i.e., slowly vary the driving frequencies $\omega_i(t)=\nu_i+\alpha_i t$, assuming linear frequency chirp for simplicity. Hence, at $t=0$, our system passes the linear resonance. Hamiltonian (20) yields the following evolution equations of our driven system near the equilibrium:

$$\dot{I}_i = \epsilon_i \sqrt{\frac{I_i}{2\nu_i}} \sin \psi_i, \tag{22}$$

$$\dot{\psi}_1 = aI_1 + bI_2 - \alpha_1 t + \frac{\epsilon_1}{2\sqrt{2\nu_1 I_1}} \cos \psi_1, \tag{23}$$

$$\dot{\psi}_2 = bI_1 + cI_2 - \alpha_2 t + \frac{\epsilon_2}{2\sqrt{2\nu_2 I_2}} \cos \psi_2. \tag{24}$$

Note that in contrast to the fully nonlinear case [see Eqs. (4) and (5)], here we keep the driving terms in Eqs. (23) and (24) because of the smallness of the actions in the vicinity of equilibrium. In fact, these are the terms, which lead to the phase locking in the system in the initial (linear) excitation stage, where one can neglect nonlinear frequency shifts aI_1+bI_2, bI_1+cI_2 in the phase mismatch equations. In this stage, the problem separates into two independent linear oscillator systems, where the phase locking is established prior passage of the linear resonances [20].

Next, in seeking slow autoresonant solution, we separate average and oscillating components in Eqs. (23) and (24), yielding a slowly varying quasisteady state described by [compare to Eqs. (12) in the fully nonlinear case]

$$a\bar{I}_1 + b\bar{I}_2 + \frac{\epsilon_1 s_1}{2\sqrt{2\nu_1 \bar{I}_1}} = \alpha_1 t \equiv \Delta\omega_1, \tag{25}$$

$$b\bar{I}_1 + c\bar{I}_2 + \frac{\epsilon_2 s_2}{2\sqrt{2\nu_2 \bar{I}_2}} = \alpha_2 t \equiv \Delta\omega_2, \tag{26}$$

where, as before, $s_i=\cos\bar{\psi}_i \approx \pm 1$. Let us consider this quasisteady state in more detail. We desire growing solutions for $\bar{I}_{1,2}$ as t goes from large negative to large positive values (passing through the linear resonance at $t=0$). Assuming that one proceeds (at large negative t) in equilibrium and that the actions are sufficiently small in the initial excitation stage, our initial quasisteady state in this stage is given by

$$\frac{\epsilon_i}{2\sqrt{2\nu_i \bar{I}_i}} s_i = \alpha_i t. \tag{27}$$

Therefore, if phase locking is established in the initial excitation stage ($t\to-\infty$), s_i must have signs opposite to that of α_i. We have seen in Sec. II that the stability of the fully nonlinear autoresonant stage implies $s_1 s_2$ and $D=ac-b^2$ having the same signs. Consequently, Eq. (27) implies that $\alpha_1\alpha_2$ must have the same sign as D. For example, when $D<0$, one must chirp the driving frequencies in opposite directions.

In the developed autoresonant stage (asymptotic at large positive t), we assume that the actions are large enough to rewrite Eqs. (25) and (26) as

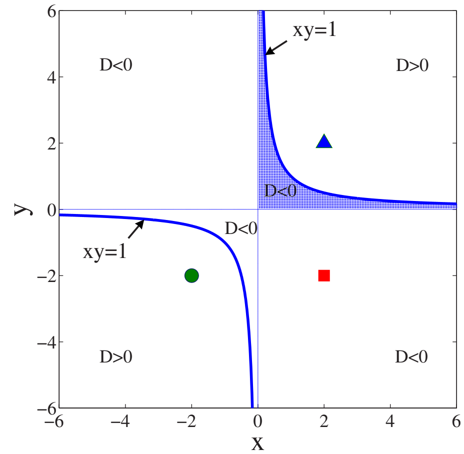

FIG. 2. (Color online) Autoresonant domain and classification by the sign of the determinant D in (x,y) space. $D>0$ $(D<0)$ correspond to $xy>1$ $(xy<1)$. The double autoresonance is forbidden in the shaded region $\{x,y>0$ and $xy<1\}$ only. The three points (▲, ■, ●) correspond to the systems studied numerically for autoresonance threshold (see Fig. 3).

$$a\bar{I}_1 + b\bar{I}_2 = \Delta\omega_1, \quad b\bar{I}_1 + c\bar{I}_2 = \Delta\omega_2.$$

This system yields linear time dependence of the autoresonant state

$$\bar{I}_1 = \frac{c\alpha_1 - b\alpha_2}{D} t, \quad \bar{I}_2 = \frac{a\alpha_2 - b\alpha_1}{D} t.$$

Furthermore, we obtain

$$\Delta\omega_1\Delta\omega_2 = (x\bar{I}_1 + \bar{I}_2)(\bar{I}_1 + y\bar{I}_2)b^2, \tag{28}$$

where, as before, $x=a/b$, $y=c/b$. Let us discuss the region in the (x,y) plane, where Eq. (28) can be satisfied, subject to $I_{1,2}>0$ and $\Delta\omega_1\Delta\omega_2>0$ (<0) for $D>0$ $(D<0)$. We analyze the four quarters of the (x,y) plane separately. (a) In the region $x,y>0$ for $I_{1,2}>0$, the right-hand side of Eq. (28) is positive. Therefore, we must have $\Delta\omega_1\Delta\omega_2>0$, and the autoresonant state exists for $D>0$ $(xy>1)$ only. In contrast, for $0<xy<1$, the autoresonance is impossible. (b) For $x<0$, $y>0$, D is always negative $(xy<1)$, so we set $\Delta\omega_1\Delta\omega_2<0$ in the left-hand side of Eq. (28) and observe that $I_2>-xI_1$ in this case (c) $x>0$, $y<0$. In this quadrant D is again negative and the autoresonant solution satisfies $I_2>-\frac{1}{y}I_1$. Finally, in case (d) $x,y<0$, we may have two scenarios. For $xy>1$, one has $-x>-\frac{1}{y}$ and condition $\Delta\omega_1\Delta\omega_2>0$ is fulfilled for $-xI_1>I_2>-\frac{1}{y}I_1$. Similarly, if $xy<1$, we have $-x<-\frac{1}{y}$, so we have either $I_2>-\frac{1}{y}I_1$ or $I_2<-xI_1$. We summarize this analysis by showing the region of existence of autoresonant solutions in the (x,y) parameter space (defined by the Hessian of the weakly nonlinear Hamiltonian) in Fig. 2.

Next, we discuss the adiabatic autoresonant state of the system as a point in the $(\bar{I}_1,\bar{I}_2)$ plane. For $\bar{I}_{1,2}$ growing linearly in time (large negative times), the quasisteady state actions move in the $(\bar{I}_1,\bar{I}_2)$ plane along a straight line of slope

$$\rho \equiv \frac{\bar{I}_2}{\bar{I}_1} = \frac{a\alpha_2 - b\alpha_1}{c\alpha_1 - b\alpha_2}. \tag{29}$$

Remarkably, one can find driving conditions such that the evolution in the $(\bar{I}_1,\bar{I}_2)$ plane in the small amplitude autoresonant regime (large negative times) would follow the same straight line. Indeed, given Hessian elements a, b, and c, we choose slope ρ such that the line $\bar{I}_2=\rho\bar{I}_1$ lies in the autoresonant region defined above via the set of parameters x, y. Next, we choose the chirp rates ratio to be

$$v \equiv \frac{\alpha_1}{\alpha_2} = \frac{a+b\rho}{b+c\rho}. \tag{30}$$

Then, the choice of the ratio of the driving amplitudes

$$u \equiv \frac{\epsilon_1}{\epsilon_2} = |v|\sqrt{\frac{w}{\rho}}, \tag{31}$$

where $w=\frac{\nu_1}{\nu_2}$, guaranties satisfaction of the adiabatic resonance condition along the straight line in the $(\bar{I}_1,\bar{I}_2)$ plane with the same slope ρ throughout the whole weakly nonlinear evolution [see Eqs. (27)]. This completes our discussion of the slow, weakly nonlinear autoresonant quasisteady state. Existence of such a state is not sufficient for successful autoresonance in the system, since we must also address the problem of its stability.

The stability of autoresonance in the vicinity of equilibrium can be approached similarly to that discussed for the fully developed autoresonance in Sec. II. Indeed, by assuming $\cos\psi_i \approx \pm 1$ throughout the weakly nonlinear excitation process, we can rewrite our evolution equations (22)–(24) as

$$\dot{I}_i = \epsilon_i\sqrt{I_i/2\nu_i}\sin\psi_i, \dot{\psi}_i = \Omega_i' - \alpha_i t, \tag{32}$$

where $\Omega_1'=aI_1+bI_2+\frac{\epsilon_1 s_1}{2\sqrt{2\nu_1\bar{I}_1}}$ and $\Omega_2'=bI_1+cI_2+\frac{\epsilon_2 s_2}{2\sqrt{2\nu_2\bar{I}_2}}$. These equations differ from the general system of evolution equations (6) discussed previously by a particular choice of the nonlinear frequency shifts $\Omega_{1,2}'$. Then, by previous analysis (see Sec. II), we conclude that if the autoresonant quasisteady state exists near the equilibrium as discussed above, it will be stable, provided, the driving amplitudes ϵ_i are sufficiently large for a given set of chirp rates α_i. One can relax this strong condition by asking the question of existence of the smallest (critical) values of the driving amplitudes still allowing transition to autoresonance by starting in equilibrium and passing through linear resonances in the system. In the case of the one-dimensional autoresonance, this question leads to the problem of thresholds [21], Here, in the two-dimensional case, we will not study this problem in the most general sense, but focus on a simpler problem of finding the critical driving amplitudes for autoresonance, when moving along a straight line $\bar{I}_2/\bar{I}_1=\rho$, as described above. In this case the ratios $u=\epsilon_1/\epsilon_2$ and $v=\alpha_1/\alpha_2$ are prescribed [see Eqs. (30) and (31)] and the problem of critical driving amplitude reduces to that of, finding critical ϵ_1^{cr} versus α_1. Let us show that this ϵ_1^{cr} scales as $\alpha_1^{3/4}$. We proceed from the original

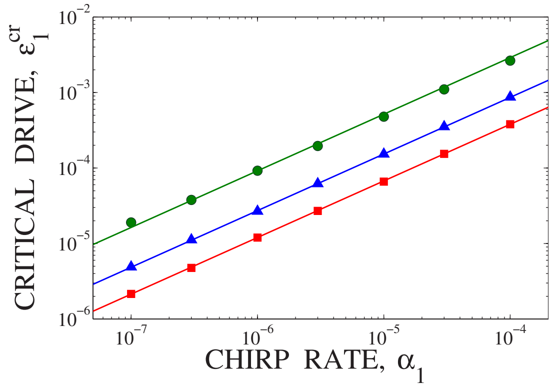

FIG. 3. (Color online) Threshold driving amplitude versus chirp rate. Each calculation corresponds to a given set $S=\{a,b,c,\nu_1,\nu_2,\rho\}$ of parameters. The straight lines correspond to best fit of slope 3/4. ▲, $S=\{2,1,2,1.6,1,1\}$. ■, $S=\{1,0.5,-1,0.25,0.475,3\}$. ●, $S=\{-1,0.5,-1,2.5,2,1\}$. These three different sets exemplify three points located in different quadrants in (x,y) parameter space shown in Fig. 2.

system (22)–(24) of weakly nonlinear evolution equations and rewrite it as

$$\frac{dI_1'}{d\tau} = \mu\sqrt{I_1'}\sin\psi_1, \quad \frac{dI_2'}{d\tau} = \mu\frac{\sqrt{w}}{u}\sqrt{I_2'}\sin\psi_2,$$

$$\frac{d\psi_1}{d\tau} = aI_1' + bI_2' - \tau + \frac{\mu}{2\sqrt{I_1'}}\sin\psi_1,$$

$$\frac{d\psi_2}{d\tau} = bI_1' + cI_2' - \frac{\tau}{v} + \frac{\mu\sqrt{w}}{2u\sqrt{I_1'}}\sin\psi_2,$$

where we have rescaled the time $\tau \equiv \alpha_1^{1/2}t$ (assuming $\alpha_1>0$ for definiteness) and actions $I_i' \equiv \alpha_1^{-1/2}I_i$, and defined parameter $\mu \equiv \frac{\epsilon_1}{\sqrt{2\nu_1}}\alpha_1^{-3/4}$. Our initial conditions (at $\tau=-\infty$) are $I_i'=0$ and $\psi_i=0$ or π (the system phase-locks prior reaching the linear resonance [20]). Therefore, μ is the only parameter controlling the asymptotic state of the system at $\tau=+\infty$, i.e., after passage through the linear resonance. Then, if there exists a minimal (critical) value of μ separating autoresonant (phase-locked) and nonautoresonant regimes, the corresponding critical driving amplitude is $\epsilon_1^{\mathrm{cr}}=\sqrt{2\nu_1}\mu^{\mathrm{cr}}\alpha_1^{3/4}$. We have checked this prediction numerically and present the results in Fig. 3. Three systems described by different parameter sets $S=\{a,b,c,\nu_1,\nu_2,\rho\}$ were considered in simulations, applying separate independent drives to each of the two degrees of freedom. We have calculated the thresholds using seven different chirp rates, α_1, covering three decades. The predicted scaling $\epsilon_1^{\mathrm{cr}} \sim \alpha_1^{3/4}$ is clearly seen in the figure.

IV. FURTHER EXAMPLES

In this section we illustrate further capabilities of our approach in forming autoresonant, nearly periodic states by presenting a number of examples outside the class of 2D problems considered above. In particular, we will discuss autoresonance in a three-particle Toda problem, a driven particle in a double well potential, and the autoresonant 3D oscillator.

A. Periodic three-particle Toda systems

The unperturbed Hamiltonian in our first example describes the famous (integrable) periodic three-particle Toda lattice [22]

$$H_0 = \frac{1}{2}\sum_{n=1}^{3} P_n^2 + \sum_{n=1}^{3} e^{Q_n - Q_{n+1}} - 3, \tag{33}$$

where $Q_4 = Q_1$. In the case of zero total momentum, by simple canonical transformation and rescaling, this problem reduces to two degrees of freedom, governed by the Hamiltonian [19]

$$\bar{H}_0 = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{24}(e^{2q_2+2\sqrt{3}q_1} + e^{2q_2-2\sqrt{3}q_1} + e^{-4q_2}) - \frac{1}{8}. \tag{34}$$

However, in contrast to the cases studied in Sec. III, this 2D problem is linearly degenerate. This yields a number of complications in applying autoresonant excitation approach to the problem. Indeed, for analyzing independent excitation of the two degrees of freedom, one needs formal transformation to AA variables in the linear limit, as described in Sec. III. But, in this limit, the choice of AA variables in degenerate problems is not unique. On the other hand, if the problem is integrable and the nonlinearity removes the degeneracy, there should exist only one set of AA variables. This nontrivial (proper) set of AA variables only should be used in the linear limit in analyzing autoresonant excitation in linearly degenerate systems. Additional complication is that the weakly nonlinear Hamiltonian of form (20) can not be calculated by using the standard canonical perturbation theory in linearly degenerate systems, since the degeneracy leads to vanishing denominators in the perturbation series. The issues of choosing the proper linear limit of the AA variables and calculating the weakly nonlinear Hamiltonians expressed in the action variables, as well as the associated problem of integrability in 2D linearly degenerate systems were addressed recently [23]. It was shown that the transformation to proper AA variables in the linear limit of Eq. (34) is given by

$$q_1 = a_1 \cos\theta_1 + a_2 \cos\theta_2, \quad q_2 = a_1 \sin\theta_1 - a_2 \sin\theta_2,$$

where $a_{1,2} \sim \sqrt{I_{1,2}}$. Then, by writing the driving term in the Hamiltonian as $H_d = q_1 f_1 + q_2 f_2$, with $f_{1,2}$ representing the driving forces, one can guess the desired form of the driving, i.e.,

$$f_1 = \varepsilon_1 \cos\varphi_1 + \varepsilon_2 \cos\varphi_2, \quad f_2 = \varepsilon_1 \sin\varphi_1 - \varepsilon_2 \sin\varphi_2, \tag{35}$$

where $\varphi_i = \int \omega_i(t)dt$, allowing excitation of each of the degrees of freedom separately. Indeed, in the resonant approximation, the driving term in the Hamiltonian with this type of forcing becomes

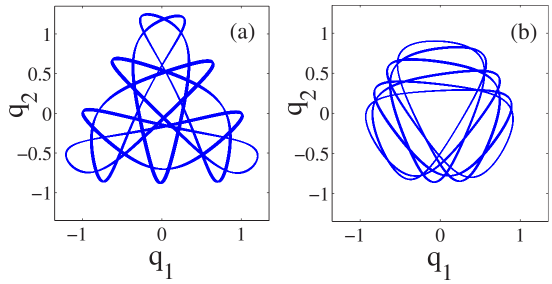

FIG. 4. (Color online) Nearly periodic states in Toda systems. (a) Autoresonant trajectory in 2D driven, reduced Toda potential with final frequencies ratio $\omega_1:\omega_2 = 5:6$. (b) Nonintegrable sixth order Taylor expanded Toda potential case with the same drives as in (a). Both trajectories contain 200 driving periods.

$$H_d \approx \varepsilon_1 a_1 \cos(\theta_1 - \varphi_1) + \varepsilon_2 a_2 \cos(\theta_2 - \varphi_2).$$

Figure 4(a) shows the numerical example of autoresonantly driven, nearly periodic trajectory obtained by using two drives of form (35) having frequencies $\omega_{1,2}$ passing through the two linear resonances at $\nu_{1,2} = 1$. The final frequencies in this simulation were $\omega_{1,2}^f = 1.44, 1.728$, yielding the final driven nearly periodic trajectory with $\omega_1:\omega_2 = 5:6$. The driving amplitudes in the simulation were $\epsilon_{1,2} = 0.0008$. The trajectory in the figure corresponds to 200 periods of driving oscillations. Note that the width of the trajectory in the figure does not indicate an instability, but similarly to other examples, is due to small and stable autoresonant oscillations around the average trajectory, as discussed in Sec. II.

The Toda problem discussed above is integrable and, therefore, the unperturbed Hamiltonian depends on the actions only. In our next example, we will consider a driven approximate Toda system obtained by Taylor expanding the exact Hamiltonian (34) to sixth order in coordinates. The resulting Hamiltonian is

$$\tilde{H}_0 = \frac{p_1^2 + p_2^2}{2} + \frac{q_1^2 + q_2^2}{2} + q_1^2 q_2 - \frac{q_2^3}{3} + \frac{(q_1^2 + q_2^2)^2}{2} + q_1^4 q_2 + \frac{2}{3}q_1^2 q_2^3 - \frac{q_2^5}{3} + \frac{q_1^6}{5} + q_1^4 q_2^2 + \frac{q_1^2 q_2^4}{3} + \frac{11}{45}q_2^6. \tag{36}$$

After the expansion, the problem loses its global integrability but, if the excitations of the system are such that the difference $H_1 = \bar{H}_0 - \tilde{H}_0$ is sufficiently small or rapidly oscillating, we still expect autoresonance in the driven system, as discussed at the beginning of Sec. II. We show such a driven, autoresonant trajectory in the sixth order truncated Toda Hamiltonian case (36) in Fig. 4(b). We used the same initial conditions, driving terms and other parameters in this simulation as in Fig. 4(a). One observes topological similarity of the trajectories in Figs. 4(a) and 4(b) but the actual trajectories in the integrable and nonintegrable cases are visibly different.

Finally, the drives in all our autoresonant examples are small perturbations and because of the driving, the autoresonant trajectories are periodic only in average. Nevertheless, if interested in finding a true periodic orbit in the unperturbed system, one can use the autoresonant trajectory as a first

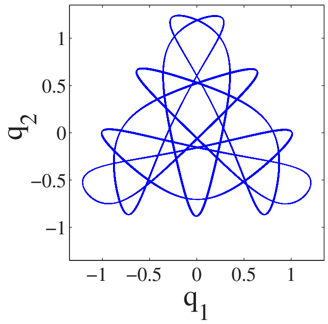

FIG. 5. (Color online) Periodic trajectory of the unperturbed three-particle Toda Hamiltonian obtained by Newton's method after four iterations, starting from the autoresonant state. The trajectory contains 200 cycles of oscillations.

guess in the unperturbed problem and calculate the true unperturbed periodic orbit by iterations, using the Newton's method [24]. An example of such calculation for the integrable Toda lattice is shown in Fig. 5, where a periodic orbit with the same rational ratio of frequencies as in Fig. 4 is shown after just three iterations of the Newton method. No visible change in this trajectory is seen in the figure after 200 periods of oscillations, indicating the high accuracy of the obtained periodic orbit.

B. Double-well potential

All our previous examples dealt with 2D potentials $V(q_1,q_2)$ having stable equilibria at $q_{1,2}=0$. How one can use autoresonance in forming periodic orbits for more complex confining potentials? The combination of the adiabatic switching [2] and autoresonance approaches yields a solution to some of these more complicated problems. This idea is illustrated in our next example. The goal was to autoresonantly excite a high energy periodic state in a double well potential $V_1=\frac{1}{2}(-3\nu_1^2q_1^2+\nu_2^2q_2^2)+(q_1^2+q_2^2)^2$, having a maximum at the origin and two minima at $(q_1,q_2)=\left(\pm\frac{\sqrt{3}}{2}\nu_1,0\right)$. The goal was achieved by simulating autoresonant dynamics in a transient potential $V(q_1,q_2,\lambda)=V_0+\lambda(V_1-V_0)$, where $V_0=\frac{1}{2}(\nu_1^2q_1^2+\nu_2^2q_2^2)+(q_1^2+q_2^2)^2$ had a single minimum at the origin and λ was a time varying parameter. In the first excitation stage, we kept $\lambda=0$, started at the origin, and applied two small, slowly chirped frequency drives, such that the driving frequencies ω_i slowly passed the linear resonances reaching some rational ratio $\omega_1:\omega_2=m:n$ in the process of evolution. This allowed excitation of the autoresonant, nearly periodic state as described above. In the following excitation stage, we continued driving the system using these fixed final frequencies and, as the phase locking continued, adiabatically varied our parameter λ from 0 to 1. At the end, this process guaranteed arrival at the original potential V_1, while continuing autoresonance in the system yielded a final periodic state with the $m:n$ periodicity. Figure 6 shows a final 4:5 resonant trajectory in the double well potential V_1 obtained via the combined autoresonant-adiabatic switching approach. The linear frequencies in this example were $\nu_{1,2}=1,1.4$, the final driving frequencies $\omega_{1,2}^f=2,2.5$, and the

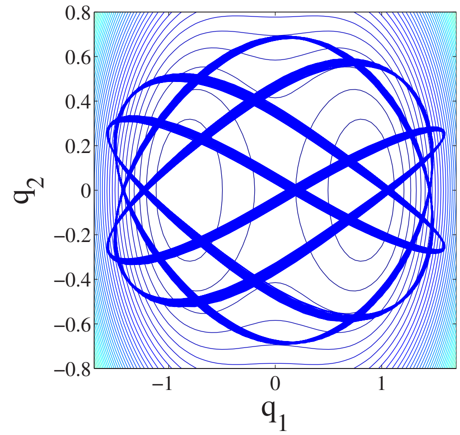

FIG. 6. (Color online) Nearly periodic orbits in a driven 2D double-well potential system. The driving frequencies $\omega_{1,2}$ were swept through the linear frequencies, reaching final values having $\omega_1:\omega_2=4:5$. This orbit contain 63 driving cycles. The constant potential contour lines are shown in the background.

driving amplitudes $\epsilon_{1,2}=0.02$. The contour lines of the final potential, V_1, are also shown in the background in the figure.

C. Three coupled oscillators

Our last example illustrates the possibility of autoresonantly controlling a higher dimensional system. A 3D oscillator governed by the Hamiltonian

$$H_0=\sum_{i=1}^{3}\left(\frac{p_i^2+\nu_i^2q_i^2}{2}+\gamma_iq_i^4\right)+\frac{\delta}{2}\sum_{i\neq j}q_i^2q_j^2 \tag{37}$$

was considered as an example. We have formed nearly periodic autoresonant state in this system by applying a superposition of three drives, i.e., had $H_1=\sum_{i=1}^{3}\epsilon_iq_i\cos\int\omega_i(t)dt$, with slowly varying driving frequencies ω_i. A direct search of proper initial conditions for this orbit is time consuming because of the six dimensionality of the phase space involved. Nevertheless, the autoresonant excitation of such 3D, nearly periodic states by three chirped frequency drives required an increase of the computational effort by 50% only, as compared to the 2D examples above. We illustrate our results in Fig. 7, showing a driven, $\omega_1:\omega_2:\omega_3=2:3:5$ resonant trajec-

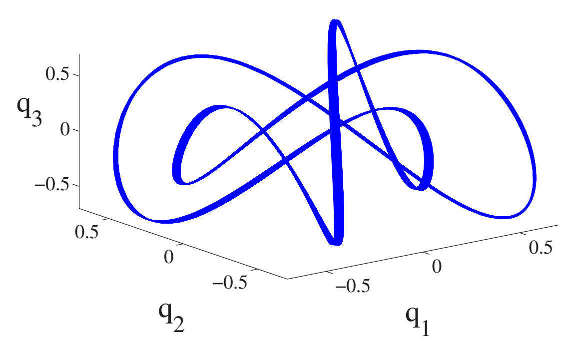

FIG. 7. (Color online) Autoresonant, nearly periodic trajectory in a three coupled oscillator system. The final frequencies ratio is $\omega_1:\omega_2:\omega_3=2:3:5$. 800 driving cycles are shown.

tory obtained by passage through the linear frequencies ν_i in the problem. The parameters in this simulation were $\nu_{1,2,3} = \{1.4, 2.4, 4.4\}$, $\gamma_{1,2,3} = \{1, 2, 4\}$, $\delta = 1$, and the final driving frequencies $\omega_{1,2,3} = \{2, 3, 5\}$. Note that the autoresonant periodic state in Fig. 7, as some of those 2D cases above, were formed in a globally nonintegrable system. We interpreted the successful arrival at the autoresonant state in Fig. 7 by using chirped frequency drives (inevitably passing through other resonances in the system) as the indication that the associated additional resonant terms in the Hamiltonian were sufficiently weak in this system.

V. CONCLUSIONS

(a) We have studied the problem of two-frequency autoresonance in 2D driven Hamiltonian dynamical systems. The essence of the autoresonance phenomenon in this case is a continuing double phase locking with two external driving perturbations, which is preserved despite a slow variation of parameters in the unperturbed Hamiltonian and/or the driving frequencies.

(b) We have assumed the existence of a local approximate action-angle (AA) representation in our system in studying the autoresonance. It was shown that in the autoresonant state, the actions evolve to satisfy the exact resonance condition in average, but also performs small amplitude oscillations around the exact resonance with characteristic frequencies scaling as a square root of the amplitudes of the driving perturbations.

(c) We have discussed the general problem of two-dimensional autoresonance in the vicinity of a local equilibrium. Passage through the nondegenerate linear resonances in this weakly nonlinear problem yielded a convenient approach to initial phase locking and autoresonance in the system, a necessary step for entering a stable, strongly nonlinear autoresonance regime. We have classified possible stable autoresonant states in the system near the equilibrium in the parameter space defined by the Hessian matrix of the unperturbed Hamiltonian in action variables.

(d) The formation of periodic trajectories in 2D systems is suggested as one of the applications of the autoresonance phenomenon. This approach may be useful in semiclassical description of multidimensional dynamical systems. We have tested this approach numerically in linearly nondegenerate 2D systems with a single minimum. Because of the smallness of the driving perturbations, nearly periodic autoresonant trajectories can be used as a first guess for finding truly closed orbits of the unperturbed system via the Newton's method.

(e) We have also discussed more complicated examples of autoresonant excitation of nearly periodic states, including a three-particle Toda system, a particle in a 2D double-well potential, and a 3D oscillator. A combined, autoresonance-adiabatic switching approach can be used to overcome difficulties of continuing autoresonance through singular regions in phase space.

ACKNOWLEDGMENT

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